Parametric nonfeedback resonance in period doubling systems

A. N. Pisarchik[1,2] and R. Corbalán[1]
[1]Departament de Física, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
[2]Stepanov Institute of Physics, National Academy of Sciences of Belarus, Skaryna Avenue 70, 220072 Minsk, Belarus
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Slow periodic modulation of a control parameter in a period doubling system leads to an interaction between stable and unstable periodic orbits. This causes a resonance in the system response at the modulation frequency. The conditions for this resonance are studied through numerical simulations of quadratic map and laser equations. The results are confirmed by experiments in a CO_2 laser with modulated losses.
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I. INTRODUCTION

The existence of nonlinear resonances with external forces is one of the characteristic properties of a dynamical system. Close to resonance the system is most sensitive to a parameter change, and even a small perturbation can lead to strong qualitative changes in the structure of phase space. Therefore, the problem of finding nonlinear resonances is very important in nonlinear dynamics and is closely related to the problem of optimal control of dynamical systems [1].

Among the modern methods for controlling dynamics, nonfeedback methods are being developed intensively [2] for nonautonomous systems as a counterpart to feedback methods [3]. It is common for the former methods to add parameter perturbations at a frequency resonant with a driving force [4]. Recently, a new approach to the nonfeedback control has been applied. It has been shown that unstable periodic orbits (UPOs) can be stabilized by a *low-frequency* (LF) modulation of a control parameter [5–7]. The term "low" means that the control frequency is much lower than a characteristic frequency of the system, for instance, the frequency of relaxation oscillations. However, the LF can drastically change dynamics of phase trajectories of a dissipative system. This is particularly remarkable when the parameter modulation forces the system to cross back and forth a bifurcation point. This periodic modulation can track an unstable periodic orbit into a parameter range where the orbit is inherently unstable [5,6]. More recently, it has been discovered that the LF control modulation at certain amplitudes and frequencies can stabilize UPOs even when the amplitude of the control is not large enough to cross static bifurcation boundaries [7]. However, such a stabilization effect is achieved only in a system where two independent parameters (for example, losses and the gain factor in a laser) are modulated with incommensurate frequencies. In this work we will study how stability properties of a dynamical system change under the influence of the small-amplitude LF modulation, if only one parameter is modulated, so that UPOs cannot be stabilized.

A related problem was recently considered by Schuster *et al.* [8] in a chaotic system with feedback parameter modulation. The feedback parameter modulation causes a drift of the system trajectory in phase space towards a UPO and back, leading to a resonant interaction with a UPO close to the actual system trajectory. They have shown that if the parameter change is chosen to be proportional to the difference between any point on the attractor and the actual trajectory, the system resonates with the most stable UPOs close to this point. We suppose that a nonfeedback parameter modulation should cause a similar effect. Since the parametric *feedback* resonance exists, the following question, which we address in this paper, naturally arises: Under which conditions does a *nonfeedback* parameter modulation cause a resonance in the system response, even if it is not initially in a chaotic state?

In this work we study conditions for the appearance of the LF parametric nonfeedback resonance in period doubled systems modeled by a quadratic map and differential equations. We investigate how the location of this resonance and its amplitude depend on the frequency and amplitude of the control modulation as well as on other system parameters. The numerical results are compared then with experiments on a loss-modulated CO_2 laser in which an additional slow loss modulation is introduced. We show that the parametric nonfeedback resonance, much as the feedback resonance [8], appears as a consequence of an interaction between stable and unstable periodic orbits that leads to growing the largest (negative) Lyapunov exponent in a system operating in a period doubling range.

The parametric resonance to be studied in this work should be distinguished from such a well-known phenomenon as the stochastic resonance [9] that was a subject of extensive investigations during recent years. The essence of the latter phenomenon is that a weak LF periodic signal which is undetectable in the absence of noise can force a bistable system to switch periodically between its two states in the presence of an optimal noise. A common point of stochastic resonance and the parametric resonance to be studied in this work is that both resonances occur as a result of a superposition of two system states. However, in the case of stochastic resonance, both states are stable, whereas in our case one of the states is unstable. Moreover, the parametric resonance can appear even in a monostable system described by the simplest iterative map without any noise.

The paper is organized as follows. First, we consider a parametrically modulated quadratic map, then we analyze

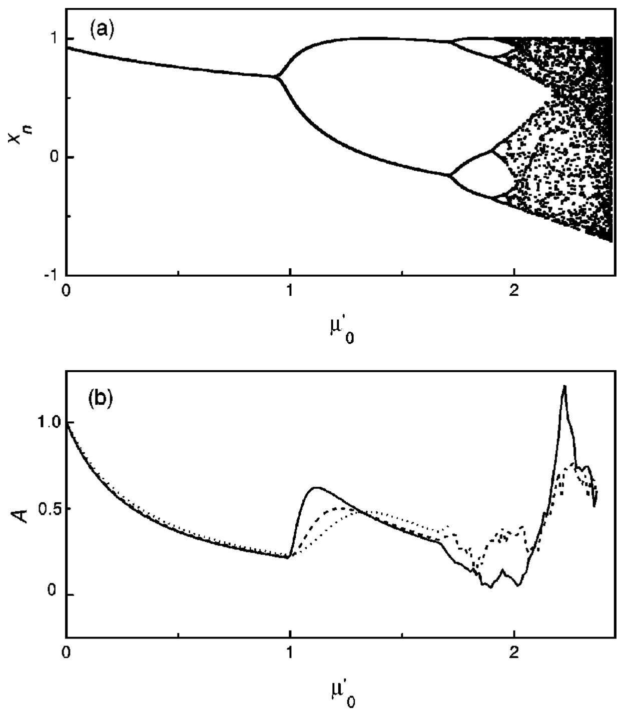

FIG. 1. (a) Bifurcation diagram of the quadratic map Eq. (1) without parameter modulation. (b) Parametric resonances at the low-frequency spectral component when the modulation is applied with the amplitude $\alpha=0.1$ and the frequencies (1) $f=0.01$ (solid line), (2) 0.05 (dashed line), and (3) 0.1 (dotted line).

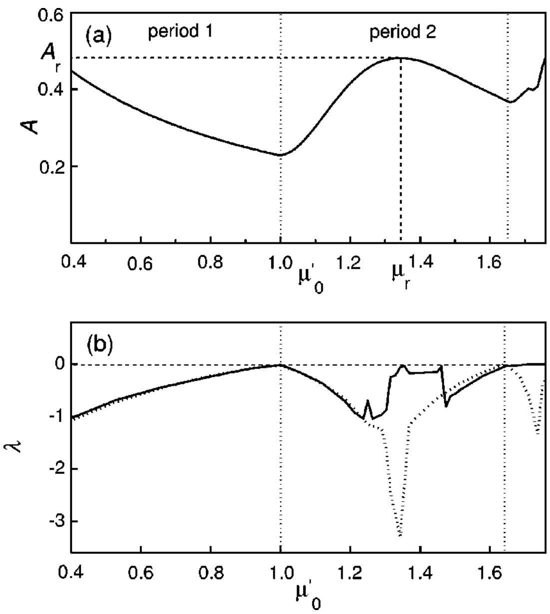

FIG. 2. (a) Normalized spectral amplitude A at $f=0.1$ versus normalized bifurcation parameter μ_0' in quadratic map Eq. (1) for $\alpha=1$. (b) Largest Lyapunov exponents in the presence (solid line) and in the absence (dotted line) of modulation. The location μ_r and the magnitude A_r of the resonance are indicated by the dashed lines. The positions of the first and second period doubling thresholds in the absence of the control modulation are shown by the vertical dotted lines. In the vicinity of the resonance, the period-2 orbit becomes more unstable because λ approaches zero.

laser equations with a modulated parameter, and finally, weprovide experimental evidence of the nonlinear parametric resonance in a loss-driven CO_2 laser.

II. QUADRATIC MAP

Consider, first, the simplest iterative map, namely, the quadratic map [10]

$$x_{n+1}=1-\mu x_n^2, \tag{1}$$

where μ is a control parameter. This dynamical system has been studied extensively in recent years [11]. The quadratic map is known to exhibit a period doubling route to chaos. For studying the influence of a slow parameter modulation, we introduce into Eq. (1) a periodic modulation of the control parameter in the form

$$\mu=\mu_0+\alpha \sin(2\pi f n), \tag{2}$$

where α and f are the modulation amplitude and frequency, μ_0 is the initial control parameter without modulation, and n is the number of the iteration or time. The additive term $\alpha \sin(2\pi fn)$ in Eq. (2) provides a LF ($f \ll 1$) modulation of the system response. Figure 1(a) shows the bifurcation diagram of Eq. (1) (without the modulation, i.e., $\alpha=0$) via the normalized bifurcation parameter $\mu_0'=\mu_0/\mu_t$, where $\mu_t=0.76$ is the threshold value of the control parameter at which the first period doubling bifurcation appears.

We study the system response by measuring the amplitude S of the f component in the Fourier spectrum of $x_n(n)$ averaged over at least 10 periods of the LF at each fixed parameter μ_0. Figure 1(b) displays the normalized spectral amplitude $A=S/S_0$ (where S_0 is the amplitude of the f-spectral component at $\mu_0=0$) versus μ_0' for different modulation frequencies. One can see that with increasing μ_0' from zero to the first period doubling bifurcation, the f component is suppressed, while above the period doubling threshold there exists a wide resonance for different modulation frequencies in the range of μ_0' correspondent to the period-2 regime. A narrow resonance is also seen in the chaotic range for very small modulation frequency. For instance, at $f=0.01$ we observe an amplification of the LF signal ($A>1$) in the chaotic range. With an increase of f, the parametric resonances reduce and broaden.

The origin of the LF resonances can be found by studying stability properties of the system under the parameter modulation. The stability of a system can be characterized by the largest Lyapunov exponent λ [12]. Figure 2(b) shows the behavior of λ versus μ_0' in the presence (solid line) and in the absence (dotted line) of the modulation with $\alpha=0.1$ and $f=0.1$. The corresponding signal response A is shown for reference in Fig. 2(a). The location and amplitude of the resonance, μ_r and A_r, are indicated in the figure. One can see that in the period-2 range, λ displays quite different behavior when the modulation is applied. The resonance in λ correlates with the resonance in A. Moreover, the location of the resonance in A coincides with the minimum in λ without the modulation. This means that the parameter modulation

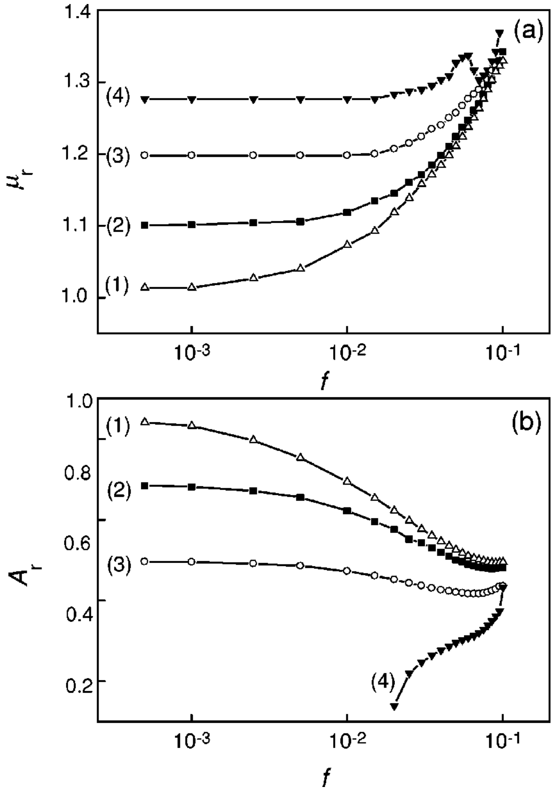

FIG. 3. (a) Location of the resonance μ_r and (b) resonance amplitude A_r versus modulation frequency for quadratic map Eq. (1) with modulation amplitudes (1) $\alpha=0.01$, (2) 0.1, (3) 0.2, and (4) 0.3. At very slow modulation the system adiabatically approaches the UPO and the resonance appears at the distance of α from the bifurcation point.

acts to ''destabilize'' the system operating in the period-2 regime. The system with the LF modulation becomes most unstable in the parameter range where it is most stable without the modulation. This indicates a strong interaction between stable and unstable periodic orbits. We find that the efficiency of this interaction depends on the amplitude and frequency of the modulation as well as on the bifurcation parameter.

Figure 3(a) shows the dependences of μ_r on the modulation frequency for different amplitudes α. One can see that at very slow (adiabatic) modulation of the control parameter, the resonance is located at a distance of α from the first period doubling threshold, i.e., $\mu_r=1+\alpha$. Thus, to obtain the resonance at very low modulation frequency, the control parameter should be changed with an amplitude equal to the difference between the actual trajectory of the period-2 orbit and the stable period-1 orbit. Only at a relatively high modulation amplitude is $\mu_r<1+\alpha$ [curve (4) in Fig. 3(a)]. In this case, the control signal cannot be considered to be small. With an increase of f, the actual trajectory deforms more and more and the position of the resonance shifts beyond from the period doubling threshold, $\mu_r>1+\alpha$, and at $f\approx 0.08$, μ_r becomes independent of α.

The dependence of the resonance amplitude A_r on the modulation frequency f is shown in Fig. 3(b) for different modulation amplitudes. One can see that A_r is higher when the signal is smaller, but with approaching f to 0.1, A_r does not depend on α, and $A_r\rightarrow 0.5$.

III. LASER EQUATIONS

Now consider a period doubling system described by a system of differential equations, for example, the simple rate-equation model of a class-B laser [13,14],

$$\frac{du}{dt}=\tau^{-1}(y-k_0-k)u, \tag{3}$$

$$\frac{dy}{dt}=(y_0-y)\gamma-uy. \tag{4}$$

Here u is proportional to the radiation density, y and y_0 are the gain and the unsaturated gain in the active medium, respectively, τ is half round-trip time of light in the resonator, γ is the gain decay rate, and k_0 is the constant part of the losses. The variable cavity losses k are modulated with two incommensurate frequencies so that

$$k=k_d\cos(2\pi f_d t)+k_c\cos(2\pi f_c t), \tag{5}$$

where f_d, k_d, and f_c, k_c are the frequencies and amplitudes of the driving and control signals, respectively. The term $k_c\cos(2\pi f_c t)$ in Eq. (5) provides a slow ($f_c\ll f_d$), small ($k_c\ll k_d$) modulation of the laser intensity. The following fixed parameters, appropriate for the experimental system described below, are used throughout our calculations: $\tau=3.5\times10^{-9}$ s, $\gamma=2.5\times10^5$ s^{-1}, $y_0=0.175$, $k_0=0.173$, $f_d=10^5$ s^{-1}, $f_c=10^4$ s^{-1}, while the parameters k_d and k_c are varied in the numerical simulations.

The bifurcation diagram of stroboscopically measured u (at each period $T=1/f_d$) shown in Fig. 4(a) displays the period doubling behavior with the bifurcation parameter k_d for the system of Eqs. (3) and (4) without the control modulation, i.e., when $k_c=0$. The influence of the slow parameter modulation is studied by measuring the amplitude S of the f_c component from the Fourier spectrum of the laser intensity at each fixed bifurcation parameter k_d. In Fig. 4(b) we plot the normalized spectral amplitude $A=S/S_0$ (where S_0 is the amplitude of the f_c-spectral component without the driving, i.e., at $k_d=0$) versus the driving amplitude k_d for two different control amplitudes, $k_c=10^{-4}$ (dots) and $k_c=10^{-5}$ (crosses). One can see that for the control frequency used in our simulations ($f_c=0.1f_d$) both the location and amplitude of the resonance do not depend on k_c. This agrees with the results obtained for the quadratic map (see Fig. 3 for $f=0.1$).

Notice that the value of A can be considered as a factor of amplification of the control signal by the laser as a nonlinear dynamical system. As can be seen from Fig. 4(b), the laser begins to amplify the LF signal ($A>1$) just after the first period doubling threshold, and A has a wide resonance with the maximum at $k_d^r=3.7$. With a further increase in k_d, the laser response decreases and the control signal is suppressed by the system, i.e., $A<1$.

Analogous to the case of the quadratic map, we study the stability of the system under the LF modulation by calculating the largest Lyapunov exponents λ (the algorithm is due

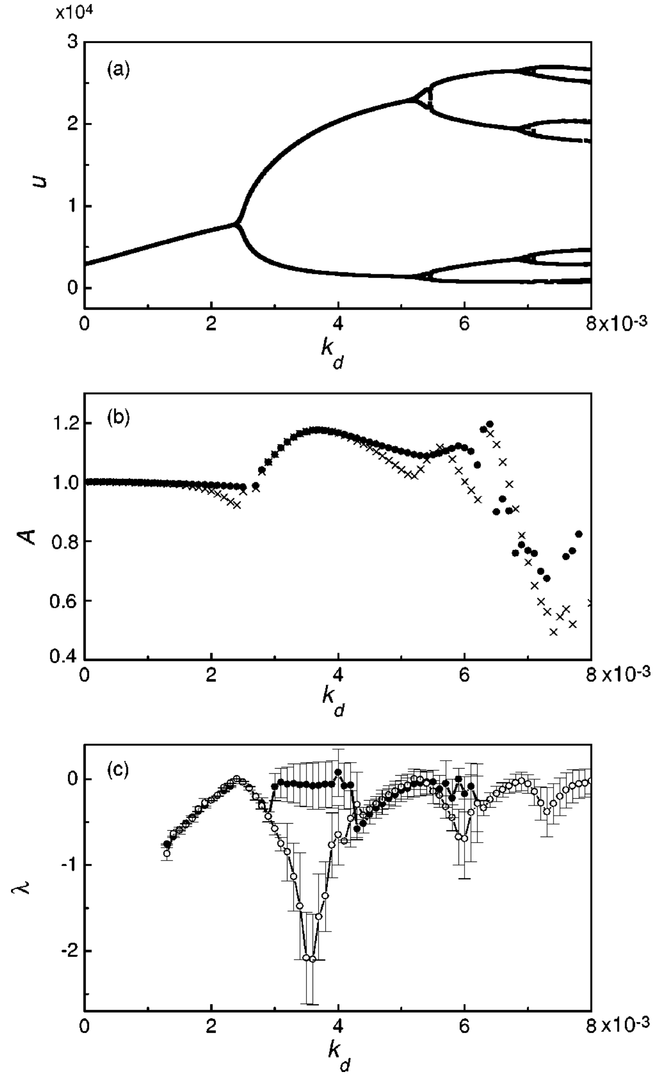

FIG. 4. (a) Bifurcation diagram of the laser Eqs. (3) and (4) with the driving amplitude k_d as a bifurcation parameter (without the control modulation, $k_c=0$). (b) Amplification factor A at the control frequency f_c versus the driving amplitude k_d for two different amplitudes of the control $k_c=10^{-4}$ (dots) and 10^{-5} (crosses). Nonlinear resonances are seen in the period-2 and period-4 ranges. (c) Largest Lyapunov exponents at the absence (open dots) and at the presence (closed dots) of the control modulation with $k_c=10^{-4}$. The slow small modulation destabilizes the system in the period-2 and period-4 ranges.

to Wolf *et al.* [12]). Figure 4(c) shows the dependences of λ on the driving amplitude in the presence (closed dots) and in the absence (open dots) of the LF modulation. One can see that the control modulation leads to ''destabilization'' of the period-2 regime approaching λ to zero. This is the reason for the resonance in the amplification factor A. By comparing Figs. 4(b) and 4(c) with Figs. 2(a) and 2(b), one can observe a close similarity in the resonant behavior of A and λ in the period doubling range for the driven laser Eqs. (3) and (4) and the parametrically modulated quadratic map Eq. (1).

Figure 5 displays the dependences of the amplification A on the control amplitude k_c in the vicinity of the resonance shown in Fig. 4(b) for different driving amplitudes. One can

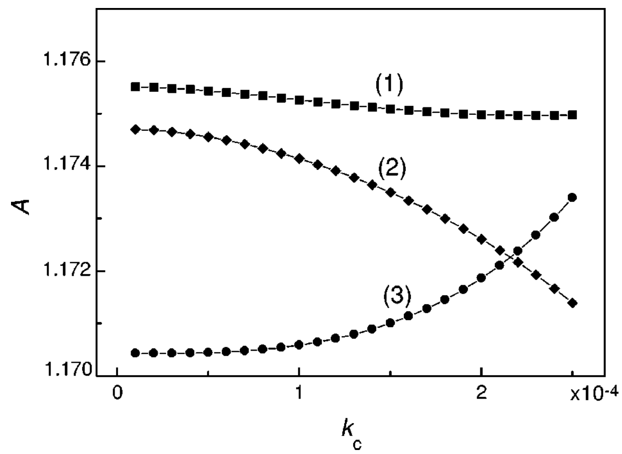

FIG. 5. (a) Amplification factor A versus the control amplitude k_c for driving amplitudes (1) $k_d=3.7$, (2) 3.6, and (3) 3.9.

see that the character of the curves changes depending on k_d with respect to the resonance. When $k_d<k_d^r$, the amplification decreases with the control amplitude k_c [curve (2)] and vice versa [curve (3)]. Only for the driving amplitude very near to k_d^r does the amplification factor almost not depend on k_c [curve (1)].

IV. EXPERIMENT

In order to check the validity of the method experimentally, we choose a single-mode CO_2 laser with modulated losses via an acousto-optic modulator. The experimental setup is similar to that described in previous works [6,7]. The electric signal $V=V_d\cos(2\pi f_d t)+V_c\cos(2\pi f_c t)$ is applied to the modulator providing time-dependent cavity losses. Here V_d and V_c are the driving and control amplitudes, respectively, $f_d=110$ kHz and $f_c=12$ kHz. The output laser intensity is detected with a $Cd_xHg_{1-x}Te$ detector and displayed on a Digital Signal Analyzer (Tektronix DSA 602A) that performs the power Fourier transform of the signal.

Figure 6 illustrates the influence of the slow control modulation at f_c on the Fourier transform spectra of the laser intensity for three different driving amplitudes correspondent to period-1 [Fig. 6(a)], period-2 [Fig. 6(b)], and period-4 [Fig. 6(c)] regimes. The spectra shown in the figure are the averaged spectra over 128 measurements. The slow parameter modulation leads also to the appearance of difference frequencies f_d+f_c and f_d-f_c. One can see that the f_c-spectral component is higher when the laser operates in the period-2 regime [Fig. 6(b)].

In Fig. 7(a) we plot the signal-to-noise ratio (SNR) at the control frequency f_c versus the driving voltage V_d. The boundaries of the period doubling and chaotic regimes (in the absence of the control modulation) are schematically indicated in the figure by the dotted lines. One can see that SNR has a resonance situated approximately at the middle part of the period-2 range. It should be noted that the control signal added to the system is relatively small as compared with the driving signal. For instance, at $V_d=1$ V and $V_c=10$ V the laser response at f_c is 100 times smaller than that at f_d. This unbalance between input and output signals

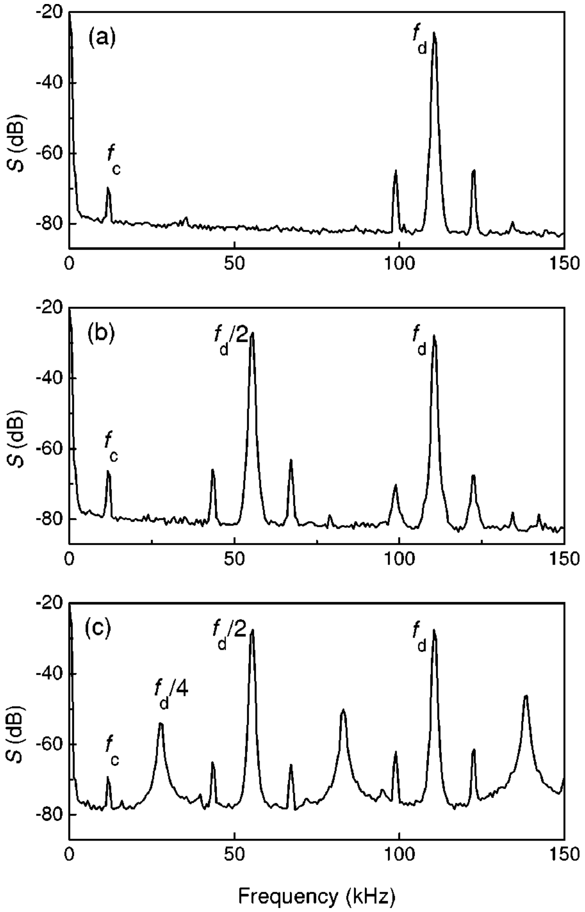

FIG. 6. Averaged experimental Fourier transform spectra of the laser intensity with the control modulation at f_c for different driving amplitudes (a) $V_d = 1.5$ V (period 1), (b) 3 V (period 2), and (c) 6 V (period 4). $f_d = 110$ kHz, $f_c = 12$ kHz, $V_c = 4$ V. The maximum in the f_c-spectral component is observed in the period-2 regime.

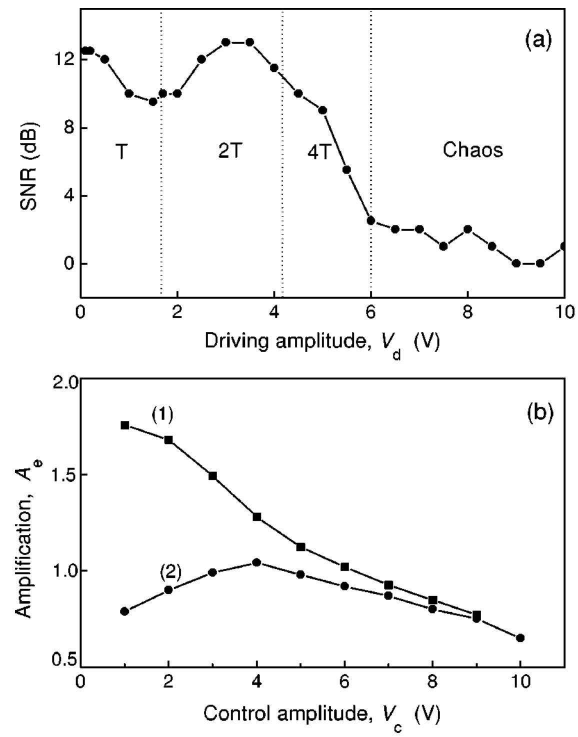

FIG. 7. (a) Experimental signal-to-noise ratio at the signal frequency taken from the averaged power spectra versus the driving amplitude. The vertical lines show the boundaries between dynamic regimes in the absence of the control modulation. (b) Experimental amplification factor versus the control amplitude at two different driving amplitudes (1) $V_d = 2.5$ V and (2) 3 V.

appears because the modulator has a strong acoustic resonance at 110 kHz, while at 12 kHz the modulator is weakly efficient.

The experimental amplification factor A_e versus the control amplitude at two different fixed values of the driving amplitude is shown in Fig. 7(b). Here A_e is defined as SNR normalized to that at $V_d = 0$. One can see that closer to the resonance shown in Fig. 7(a), the amplification factor has only a weak dependence on the control amplitude [curve (2)].

Thus, the laser operated in a period doubling regime can amplify the slow signal. The origin of this amplification is an interaction between stable and unstable periodic orbits due to a slow periodic drift of the actual system trajectory induced by the LF parameter modulation. The comparison of the experimentally obtained resonance in Fig. 7(a) with the numerical ones shown in Figs. 1(b) and 4(b) displays a good qualitative agreement. We do not consider here more complex models for dynamic description of a CO_2 laser (see, for example, [15] or [7]) that probably can provide not only qualitative but also reasonable quantitative agreement with experiments. However, the coincidence of the results obtained in the simplest systems and in the experiments is a good reason to believe that the low-frequency parametric resonance is a general phenomenon.

V. CONCLUSIONS

In conclusion, we have shown numerically with parametrically modulated quadratic map and laser equations, and experimentally in a loss-driven CO_2 laser, that a low-frequency parameter modulation in the period doubling range leads to a resonance in the system response at the low frequency. The physical mechanism underlying this effect is very similar to that in the case of the feedback parameter modulation [8]. The low frequency causes a drift of the system trajectory in phase space towards an unstable periodic orbit and back, leading, at certain parameters, to the resonant interaction with an unstable periodic orbit close to the actual system trajectory. As distinct from the chaotic system, the low frequency in a period doubled system ''decreases'' the stability of the system in the sense that the largest (negative) Lyapunov exponent grows and approaches zero at the resonance.

We have shown how the position and amplitude of the resonance depend on the amplitude and frequency of the parameter modulation. The occurrence of the low-frequency parametric resonance in different dynamical systems, especially in such a simple system as a quadratic map, induces us to believe that this is a general phenomenon and hence could be also observed in other nonlinear systems. Although we did not discover a signal amplification in the quadratic map $(A_r<1)$, the existence of amplification in the vicinity of the resonance derived from the laser equations and in experiments can be of interest for communications with the use of a period doubled system as a low-frequency signal amplifier.

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